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# The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation

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## Abstract

We discuss the relationship between the recurrence coefficients of orthogonal polynomials with respect to a semi-classical Laguerre weight and classical solutions of the fourth Painlevé equation. We show that the coefficients in these recurrence relations can be expressed in terms of Wronskians of parabolic cylinder functions which arise in the description of special function solutions of the fourth Painlevé equation.

## 1 Introduction

In this paper we are concerned with the coefficients in the three-term recurrence relations for orthogonal polynomials with respect to the semi-classical Laguerre weight

$$\omega(x; t) = x^\lambda \exp(-x^2 + tx), \quad x \in \mathbb{R}^+, \quad (1.1)$$

with parameters  $\lambda > -1$  and  $t \in \mathbb{R}$ , which has been recently studied by Boelen and van Assche [8] and Filipuk, van Assche and Zhang [23]. It is shown that these recurrence coefficients can be expressed in terms of Wronskians that arise in the description of special function solutions of the fourth Painlevé equation ( $P_{IV}$ )

$$\frac{d^2 q}{dz^2} = \frac{1}{2q} \left( \frac{dq}{dz} \right)^2 + \frac{3}{2} q^3 + 4zq^2 + 2(z^2 - A)q + \frac{B}{q}, \quad (1.2)$$

where  $A$  and  $B$  are constants, which are expressed in terms of parabolic cylinder functions.

The relationship between semi-classical orthogonal polynomials and integrable equations dates back to the work of Shohat [65] and later Freud [35]. However it was not until the work of Fokas, Its and Kapaev [27, 28] that these equations were identified as discrete Painlevé equations. The relationship between semi-classical orthogonal polynomials and the (continuous) Painlevé equations was demonstrated by Magnus [51, 52] who showed that the coefficients in the three-term recurrence relation for the Freud weight [35, 74]

$$\omega(x; t) = \exp\left(-\frac{1}{4}x^4 - tx^2\right), \quad x \in \mathbb{R}, \quad (1.3)$$

with  $t \in \mathbb{R}$  a parameter, can be expressed in terms of solutions of  $P_{IV}$  (1.2).

A motivation for this work is the fact that recurrence coefficients of semi-classical orthogonal polynomials can often be expressed in terms of solutions of the Painlevé equations. For example, recurrence coefficients are expressed in terms of solutions of  $P_{II}$  for semi-classical orthogonal polynomials with respect to the Airy weight

$$\omega(x; t) = \exp\left(\frac{1}{3}x^3 + tx\right), \quad x^3 < 0,$$

with  $t \in \mathbb{R}$  a parameter [51]; in terms of solutions of  $P_{\text{III}}$  for the perturbed Laguerre weight

$$\omega(x; t) = x^\alpha \exp(-x - t/x), \quad x \in \mathbb{R}^+,$$

with  $\alpha > 0$  and  $t \in \mathbb{R}^+$  parameters [12]; in terms of solutions of  $P_V$  for the weights

$$\begin{aligned} \omega(x; t) &= (1-x)^\alpha (1+x)^\beta e^{-tx}, & x \in [-1, 1], \\ \omega(x; t) &= x^\alpha (1-x)^\beta e^{-t/x}, & x \in [0, 1], \\ \omega(x; t) &= x^\alpha (x+t)^\beta e^{-x}, & x \in \mathbb{R}^+, \end{aligned}$$

with  $\alpha, \beta > 0$  and  $t \in \mathbb{R}^+$  parameters [3, 10, 34]; and in terms of solutions of  $P_{\text{VI}}$  for the generalized Jacobi weight

$$\omega(x; t) = x^\alpha (1-x)^\beta (t-x)^\gamma, \quad x \in [0, 1],$$

with  $\alpha, \beta, \gamma > 0$  and  $t \in \mathbb{R}^+$  parameters [20, 51].

Recurrence coefficients for orthogonal polynomials with respect to discontinuous weights which involve the Heaviside function  $\mathcal{H}(x)$  have also been expressed in terms of solutions of Painlevé equations [2, 14, 31, 33], while recurrence coefficients for orthogonal polynomials with respect to discrete weights have been expressed in terms of solutions of Painlevé equations [7, 6, 18, 21, 22].

This paper is organized as follows: in §2, we review some properties of orthogonal polynomials; in §3, we review some properties of the fourth Painlevé equation (1.2), including its Hamiltonian structure §3.1, Bäcklund and Schlesinger transformations §3.2 and special function solutions §3.3; in §4 we express the coefficients which arise in the three-term recurrence relation associated with orthogonal polynomials for the semi-classical Laguerre weight (1.1) in terms of Wronskians that arise in the description of special function solutions of  $P_{\text{IV}}$  (1.2); in §5 we derive asymptotic expansions for the recurrence coefficients; in §6 we discuss orthogonal polynomials with respect to the semi-classical Hermite weight

$$\omega(x; t) = |x|^\lambda \exp(-x^2 + tx), \quad x, t \in \mathbb{R}, \quad \lambda > -1,$$

which is an extension of the semi-classical Laguerre weight (1.1) to the whole real line, and show that the recurrence coefficients are also expressed in terms of Wronskians that arise in the description of special function solutions of  $P_{\text{IV}}$  (1.2); and in §7 we discuss our results.

## 2 Orthogonal polynomials

Let  $P_n(x)$ ,  $n \in \mathbb{N}$ , be the monic orthogonal polynomial of degree  $n$  in  $x$  with respect to a positive weight  $\omega(x)$  on  $(a, b)$ , a finite or infinite interval in  $\mathbb{R}$ , such that

$$\int_a^b P_m(x) P_n(x) \omega(x) dx = h_n \delta_{m,n}, \quad h_n > 0, \quad (2.1)$$

where  $\delta_{m,n}$  denotes the Kronekar delta. One of the most important properties of orthogonal polynomials is that they satisfy a three-term recurrence relationship of the form

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \quad (2.2)$$

where the coefficients  $\alpha_n$  and  $\beta_n$  are given by the integrals

$$\alpha_n = \frac{1}{h_n} \int_a^b x P_n^2(x) \omega(x) dx, \quad \beta_n = \frac{1}{h_{n-1}} \int_a^b x P_{n-1}(x) P_n(x) \omega(x) dx, \quad (2.3)$$

with  $P_{-1}(x) = 0$  and  $P_0(x) = 1$ . These coefficients in the three-term recurrence relationship can also be expressed in terms of determinants whose coefficients are given in terms of the moments associated with the weight  $\omega(x)$ . Specifically, the coefficients  $\alpha_n$  and  $\beta_n$  in the recurrence relation (2.2) are given by

$$\alpha_n = \frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\tilde{\Delta}_n}{\Delta_n}, \quad \beta_n = \frac{\Delta_{n+1} \Delta_{n-1}}{\Delta_n^2}, \quad (2.4)$$

where  $\Delta_n$  is the Hankel determinant

$$\Delta_n = \det [\mu_{j+k}]_{j,k=0}^n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}, \quad n \geq 1, \quad (2.5a)$$

with  $\Delta_0 = 1$ ,  $\Delta_{-1} = 0$ , and  $\tilde{\Delta}_n$  is the determinant

$$\tilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}, \quad n \geq 1, \quad (2.5b)$$

with  $\tilde{\Delta}_0 = 0$  and  $\mu_k$ , the  $k$ th moment, is given by the integral

$$\mu_k = \int_a^b x^k \omega(x) dx. \quad (2.6)$$

We remark that the Hankel determinant  $\Delta_n$  (2.5a) also has the integral representation

$$\Delta_n = \frac{1}{n!} \int_a^b \cdots \int_a^b \prod_{\ell=1}^n \omega(x_\ell) \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 dx_1 \cdots dx_n, \quad n \geq 1. \quad (2.7)$$

The monic polynomial  $P_n(x)$  can be uniquely expressed as the determinant

$$P_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}, \quad (2.8)$$

and the normalisation constants as

$$h_n = \frac{\Delta_{n+1}}{\Delta_n}, \quad h_0 = \Delta_1 = \mu_0. \quad (2.9)$$

For further information about orthogonal polynomials see, for example, [15, 43, 69].

Now suppose that the weight has the form

$$w(x; t) = \omega_0(x) \exp(xt), \quad x \in [a, b], \quad (2.10)$$

where  $t$  is a parameter, with finite moments for all  $t \in \mathbb{R}$ , which is the case for the semi-classical Laguerre weight (1.1). If the weight has the form (2.10), which depends on the parameter  $t$ , then the orthogonal polynomials  $P_n(x)$ , the recurrence coefficients  $\alpha_n, \beta_n$  given by (2.4), the determinants  $\Delta_n, \tilde{\Delta}_n$  given by (2.5) and the moments  $\mu_k$  given by (2.6) are now functions of  $t$ . Specifically, in this case then

$$\mu_k = \int_a^b x^k \omega_0(x) \exp(xt) dx = \frac{d^k}{dt^k} \left( \int_a^b \omega_0(x) \exp(xt) dx \right) = \frac{d^k \mu_0}{dt^k}.$$

Further the recurrence relation has the form

$$xP_n(x; t) = P_{n+1}(x; t) + \alpha_n(t)P_n(x; t) + \beta_n(t)P_{n-1}(x; t), \quad (2.11)$$

where we have explicitly indicated that the coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  depend on  $t$ .

**Theorem 2.1.** *If the weight has the form (2.10), then the determinants  $\Delta_n(t)$  and  $\tilde{\Delta}_n(t)$  given by (2.5) can be written as the Wronskians*

$$\Delta_n(t) = \mathcal{W} \left( \mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}} \right), \quad \tilde{\Delta}_n(t) = \frac{d\Delta_n}{dt}. \quad (2.12)$$

*Proof.* Since  $\mu_k = \frac{d^k \mu_0}{dt^k}$ , then the determinant  $\Delta_n(t)$  can be written in the form

$$\Delta_n(t) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} = \mathcal{W} \left( \mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}} \right),$$

as required, and the determinant  $\tilde{\Delta}_n(t)$ , can be written in the form

$$\begin{aligned}\tilde{\Delta}_n(t) &= \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix} = \mathcal{W}\left(\mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-2}\mu_0}{dt^{n-2}}, \frac{d^n\mu_0}{dt^n}\right) \\ &= \frac{d}{dt} \mathcal{W}\left(\mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}}\right) = \frac{d\Delta_n}{dt},\end{aligned}$$

as required.  $\square$

The Hankel determinant  $\Delta_n(t)$  satisfies the Toda equation, as shown in the following theorem.

**Theorem 2.2.** *The Hankel determinant  $\Delta_n(t)$  given by (2.12) satisfies the Toda equation*

$$\frac{d^2}{dt^2} \ln \Delta_n(t) = \frac{\Delta_{n-1}(t)\Delta_{n+1}(t)}{\Delta_n^2(t)}. \quad (2.13)$$

*Proof.* See, for example, Nakamira and Zhedanov [57, Proposition 1]; also [63, 67].  $\square$

Using Theorems 2.1 and 2.2 we can express the recurrence coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in terms of derivatives of the Hankel determinant  $\Delta_n(t)$  and so obtain explicit expressions for these coefficients.

**Theorem 2.3.** *The coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (2.11) associated with monic polynomials orthogonal with respect to a weight of the form (2.10) are given by*

$$\alpha_n(t) = \frac{d}{dt} \ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)}, \quad \beta_n(t) = \frac{d^2}{dt^2} \ln \Delta_n(t), \quad (2.14)$$

with  $\Delta_n(t)$  is the Hankel determinant given by (2.12).

*Proof.* By definition the coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (2.11) are given by

$$\alpha_n(t) = \frac{\tilde{\Delta}_{n+1}(t)}{\Delta_{n+1}(t)} - \frac{\tilde{\Delta}_n(t)}{\Delta_n(t)}, \quad \beta_n(t) = \frac{\Delta_{n-1}(t)\Delta_{n+1}(t)}{\Delta_n^2(t)},$$

where the determinants  $\Delta_n$  and  $\tilde{\Delta}_n$  are given by (2.5). Hence from (2.12)

$$\alpha_n(t) = \frac{\tilde{\Delta}_{n+1}(t)}{\Delta_{n+1}(t)} - \frac{\tilde{\Delta}_n(t)}{\Delta_n(t)} = \frac{1}{\Delta_{n+1}} \frac{d\Delta_{n+1}}{dt} - \frac{1}{\Delta_n} \frac{d\Delta_n}{dt},$$

and so

$$\alpha_n(t) = \frac{d}{dt} \ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)}, \quad (2.15)$$

as required. By definition

$$\beta_n(t) = \frac{\Delta_{n-1}(t)\Delta_{n+1}(t)}{\Delta_n^2(t)},$$

and so from Theorem 2.2 we have

$$\beta_n(t) = \frac{d^2}{dt^2} \ln \Delta_n(t), \quad (2.16)$$

as required.  $\square$

Additionally the coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (2.11) satisfy a Toda system.

**Theorem 2.4.** *The coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (2.11) associated with a weight of the form (2.10) satisfy the Toda system*

$$\frac{d\alpha_n}{dt} = \beta_{n+1} - \beta_n, \quad \frac{d\beta_n}{dt} = \beta_n(\alpha_n - \alpha_{n-1}). \quad (2.17)$$

*Proof.* See Ismail [43, §2.8, p. 41] and Moser [55]; see also [6] for further details and a direct proof in the case of the semi-classical Laguerre weight (1.1).  $\square$

Suppose  $P_n(x)$ , for  $n \in \mathbb{N}$ , is a sequence of *classical* orthogonal polynomials (such as Hermite, Laguerre and Jacobi polynomials), then  $P_n(x)$  is a solution of a second-order ordinary differential equation of the form

$$\sigma(x) \frac{d^2 P_n}{dx^2} + \tau(x) \frac{d P_n}{dx} = \lambda_n P_n, \quad (2.18)$$

where  $\sigma(x)$  is a monic polynomial with  $\deg(\sigma) \leq 2$ ,  $\tau(x)$  is a polynomial with  $\deg(\tau) = 1$ , and  $\lambda_n$  is a real number which depends on the degree of the polynomial solution, see Bochner [5]. Equivalently, the weights of classical orthogonal polynomials satisfy a first-order ordinary differential equation, the *Pearson equation*

$$\frac{d}{dx} [\sigma(x)\omega(x)] = \tau(x)\omega(x), \quad (2.19)$$

with  $\sigma(x)$  and  $\tau(x)$  the same polynomials as in (2.18), see, for example, [1, 5, 15]. However for *semi-classical* orthogonal polynomials, the weight function  $\omega(x)$  satisfies the Pearson equation (2.19) with either  $\deg(\sigma) > 2$  or  $\deg(\tau) > 1$ , see, for example, [41, 53]. For example, the Pearson equation (2.19) is satisfied for the weight (1.1) with

$$\sigma(x) = x, \quad \tau(x) = -2x^2 + tx + \lambda + 1,$$

and so the weight (1.1) is indeed a semi-classical weight function. Filipuk, van Assche and Zhang [23] comment that

*“We note that for classical orthogonal polynomials (Hermite, Laguerre, Jacobi) one knows these recurrence coefficients explicitly in contrast to non-classical weights”.*

In §4 we show that, in the case of the semi-classical Laguerre weight (1.1), the determinants  $\Delta_n(t)$  and  $\tilde{\Delta}_n(t)$  can be explicitly written as Wronskians which arise in the description of special function solutions of  $P_{IV}$  (1.2) that are expressed in terms of parabolic cylinder functions  $D_\nu(z)$  when  $\lambda \notin \mathbb{Z}$ , or error functions  $\text{erf}(z)$  when  $\lambda = n \in \mathbb{Z}$ . Consequently the recurrence coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  (2.4) associated with orthogonal polynomials for the semi-classical Laguerre weight (1.1) can also be explicitly written in terms of these Wronskians.

### 3 Properties of the fourth Painlevé equation

The six Painlevé equations ( $P_I$ – $P_{VI}$ ) were first discovered by Painlevé, Gambier and their colleagues in an investigation of which second order ordinary differential equations of the form

$$\frac{d^2 q}{dz^2} = F\left(\frac{dq}{dz}, q, z\right), \quad (3.1)$$

where  $F$  is rational in  $dq/dz$  and  $q$  and analytic in  $z$ , have the property that their solutions have no movable branch points. They showed that there were fifty canonical equations of the form (3.1) with this property, now known as the *Painlevé property*. Further Painlevé, Gambier and their colleagues showed that of these fifty equations, forty-four are either integrable in terms of previously known functions (such as elliptic functions or are equivalent to linear equations) or reducible to one of six new nonlinear ordinary differential equations, which define new transcendental functions, see Ince [42]. The Painlevé equations can be thought of as nonlinear analogues of the classical special functions [17, 26, 39, 44, 71], and arise in a wide variety of applications, for example random matrices, cf. [30, 62].

#### 3.1 Hamiltonian structure

Each of the Painlevé equations  $P_I$ – $P_{VI}$  can be written as a Hamiltonian system

$$\frac{dq}{dz} = \frac{\partial \mathcal{H}_J}{\partial p}, \quad \frac{dp}{dz} = -\frac{\partial \mathcal{H}_J}{\partial q}, \quad (3.2)$$

for a suitable Hamiltonian function  $\mathcal{H}_J(q, p, z)$  [45, 58, 60]. The function  $\sigma(z) \equiv \mathcal{H}_J(q, p, z)$  satisfies a second-order, second-degree ordinary differential equation, whose solution is expressible in terms of the solution of the associated Painlevé equation [45, 59, 60].

The Hamiltonian associated with  $P_{IV}$  (1.2) is

$$\mathcal{H}_{IV}(q, p, z; \vartheta_0, \vartheta_\infty) = 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q, \quad (3.3)$$

with  $\vartheta_0$  and  $\vartheta_\infty$  parameters [45, 58, 59, 60], and so from (3.2)

$$\frac{dq}{dz} = 4qp - q^2 - 2zq - 2\vartheta_0, \quad (3.4a)$$

$$\frac{dp}{dz} = -2p^2 + 2qp + 2zp - \vartheta_\infty. \quad (3.4b)$$

Solving (3.4a) for  $p$  and substituting in (3.4b) yields

$$\frac{d^2q}{dz^2} = \frac{1}{2q} \left( \frac{dq}{dz} \right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 + \vartheta_0 - 2\vartheta_\infty - 1)q - \frac{2\vartheta_0^2}{q}, \quad (3.5)$$

which is  $P_{IV}$  (1.2) with  $A = 1 - \vartheta_0 + 2\vartheta_\infty$  and  $B = -2\vartheta_0^2$ . Analogously, solving (3.4b) for  $q$  and substituting in (3.4a) yields

$$\frac{d^2p}{dz^2} = \frac{1}{2p} \left( \frac{dp}{dz} \right)^2 + 6p^3 - 8zp^2 + 2(z^2 - 2\vartheta_0 + \vartheta_\infty + 1)p - \frac{\vartheta_\infty^2}{2p}. \quad (3.6)$$

Then letting  $p = -\frac{1}{2}w$  yields  $P_{IV}$  (1.2) with  $A = -1 + 2\vartheta_0 - \vartheta_\infty$  and  $B = -2\vartheta_\infty^2$ .

An important property of the Hamiltonian, which is very useful in applications, is that it satisfies a second-order, second-degree ordinary differential equation.

**Theorem 3.1.** *Consider the function*

$$\sigma(z; \vartheta_0, \vartheta_\infty) = 2qp^2 - (q^2 + 2zq + 2\vartheta_0)p + \vartheta_\infty q, \quad (3.7)$$

where  $q$  and  $p$  satisfy the system (3.4), then  $\sigma$  satisfies the second-order, second-degree ordinary differential equation

$$\left( \frac{d^2\sigma}{dz^2} \right)^2 - 4 \left( z \frac{d\sigma}{dz} - \sigma \right)^2 + 4 \frac{d\sigma}{dz} \left( \frac{d\sigma}{dz} + 2\vartheta_0 \right) \left( \frac{d\sigma}{dz} + 2\vartheta_\infty \right) = 0. \quad (3.8)$$

Conversely, if  $\sigma$  is a solution of (3.8), then solutions of the Hamiltonian system (3.4) are given by

$$q = \frac{\sigma'' - 2z\sigma' + 2\sigma}{2(\sigma' + 2\vartheta_\infty)}, \quad p = \frac{\sigma'' + 2z\sigma' - 2\sigma}{4(\sigma' + 2\vartheta_0)}, \quad ' = \frac{d}{dz}. \quad (3.9)$$

*Proof.* See Jimbo and Miwa [45] and Okamoto [58, 59, 60].  $\square$

**Remarks 3.2.**

1. Equation (3.8), which is often known as  $S_{IV}$  (or the  $P_{IV}$   $\sigma$ -equation), is equivalent to equation SD-I.c in the classification of second order, second-degree ordinary differential equations with the Painlevé property by Cosgrove and Scoufis [19], an equation first derived and solved by Chazy [9].
2. Theorem 3.1 shows that solutions of equation (3.8) are in a one-to-one correspondence with solutions of the Hamiltonian system (3.4), and so are in a one-to-one correspondence with solutions of  $P_{IV}$  (1.2).
3. Equation (3.8) also arises in various applications, for example random matrix theory [32, 33, 48, 70].

### 3.2 Bäcklund and Schlesinger transformations

The Painlevé equations  $P_{II}$ – $P_{VI}$  possess *Bäcklund transformations* which relate one solution to another solution either of the same equation, with different values of the parameters, or another equation (see [17, 24, 39] and the references therein). An important application of the Bäcklund transformations is that they generate hierarchies of classical solutions of the Painlevé equations, which are discussed in §3.3.

Bäcklund transformations for  $P_{IV}$  (1.2) are given as follows.

**Theorem 3.3.** *Let  $q_0 = w(z; A_0, B_0)$  and  $q_j^\pm = w(z; A_j^\pm, B_j^\pm)$ ,  $j = 1, 2, 3, 4$  be solutions of  $P_{IV}$  (1.2) with*

$$A_1^\pm = \frac{1}{4}(2 - 2A_0 \pm 3\sqrt{-2B_0}), \quad B_1^\pm = -\frac{1}{2}(1 + A_0 \pm \frac{1}{2}\sqrt{-2B_0})^2, \quad (3.10a)$$

$$A_2^\pm = -\frac{1}{4}(2 + 2A_0 \pm 3\sqrt{-2B_0}), \quad B_2^\pm = -\frac{1}{2}(1 - A_0 \pm \frac{1}{2}\sqrt{-2B_0})^2, \quad (3.10b)$$

$$A_3^\pm = \frac{3}{2} - \frac{1}{2}A_0 \mp \frac{3}{4}\sqrt{-2B_0}, \quad B_3^\pm = -\frac{1}{2}(1 - A_0 \pm \frac{1}{2}\sqrt{-2B_0})^2, \quad (3.10c)$$

$$A_4^\pm = -\frac{3}{2} - \frac{1}{2}A_0 \mp \frac{3}{4}\sqrt{-2B_0}, \quad B_4^\pm = -\frac{1}{2}(-1 - A_0 \pm \frac{1}{2}\sqrt{-2B_0})^2. \quad (3.10d)$$

Then

$$\mathcal{T}_1^\pm : \quad q_1^\pm = \frac{q'_0 - q_0^2 - 2zq_0 \mp \sqrt{-2B_0}}{2q_0}, \quad (3.11a)$$

$$\mathcal{T}_2^\pm : \quad q_2^\pm = -\frac{q'_0 + q_0^2 + 2zq_0 \mp \sqrt{-2B_0}}{2q_0}, \quad (3.11b)$$

$$\mathcal{T}_3^\pm : \quad q_3^\pm = q_0 + \frac{2(1 - A_0 \mp \frac{1}{2}\sqrt{-2B_0})q_0}{q'_0 \pm \sqrt{-2B_0} + 2zq_0 + q_0^2}, \quad (3.11c)$$

$$\mathcal{T}_4^\pm : \quad q_4^\pm = q_0 + \frac{2(1 + A_0 \pm \frac{1}{2}\sqrt{-2B_0})q_0}{q'_0 \mp \sqrt{-2B_0} - 2zq_0 - q_0^2}, \quad (3.11d)$$

valid when the denominators are non-zero, and where the upper signs or the lower signs are taken throughout each transformation.

*Proof.* See Gromak [37, 38] and Lukashevich [50]; also [4, 39, 56].  $\square$

A class of Bäcklund transformations for the Painlevé equations is generated by so-called *Schlesinger transformations* of the associated isomonodromy problems. Fokas, Mugan and Ablowitz [29], deduced the following Schlesinger transformations  $\mathcal{R}_1$ – $\mathcal{R}_4$  for  $P_{IV}$ .

$$\mathcal{R}_1 : \quad q_1(z; A_1, B_1) = \frac{(q' + \sqrt{-2B})^2 + (4A + 4 - 2\sqrt{-2B})q^2 - q^2(q + 2z)^2}{2q(q^2 + 2zq - q' - \sqrt{-2B})}, \quad (3.12a)$$

$$\mathcal{R}_2 : \quad q_2(z; A_2, B_2) = \frac{(q' - \sqrt{-2B})^2 + (4A - 4 - 2\sqrt{-2B})q^2 - q^2(q + 2z)^2}{2q(q^2 + 2zq + q' - \sqrt{-2B})}, \quad (3.12b)$$

$$\mathcal{R}_3 : \quad q_3(z; A_3, B_3) = \frac{(q' - \sqrt{-2B})^2 - (4A + 4 + 2\sqrt{-2B})q^2 - q^2(q + 2z)^2}{2q(q^2 + 2zq - q' + \sqrt{-2B})}, \quad (3.12c)$$

$$\mathcal{R}_4 : \quad q_4(z; A_4, B_4) = \frac{(q' + \sqrt{-2B})^2 + (4A - 4 + 2\sqrt{-2B})q^2 - q^2(q + 2z)^2}{2q(q^2 + 2zq + q' + \sqrt{-2B})}, \quad (3.12d)$$

where  $q \equiv q(z; A, B)$  and

$$(A_1, B_1) = \left(A + 1, -\frac{1}{2}(2 - \sqrt{-2B})^2\right), \quad (A_2, B_2) = \left(A - 1, -\frac{1}{2}(2 + \sqrt{-2B})^2\right), \quad (3.12e)$$

$$(A_3, B_3) = \left(A + 1, -\frac{1}{2}(2 + \sqrt{-2B})^2\right), \quad (A_4, B_4) = \left(A - 1, -\frac{1}{2}(2 - \sqrt{-2B})^2\right). \quad (3.12f)$$

Fokas, Mugan and Ablowitz [29] also defined the composite transformations  $\mathcal{R}_5 = \mathcal{R}_1\mathcal{R}_3$  and  $\mathcal{R}_7 = \mathcal{R}_2\mathcal{R}_4$  given by

$$\mathcal{R}_5 : \quad q_5(z; A_5, B_5) = \frac{(q' - q^2 - 2zq)^2 + 2B}{2q\{q' - q^2 - 2zq + 2(A + 1)\}}, \quad (3.13a)$$

$$\mathcal{R}_7 : \quad q_7(z; A_7, B_7) = -\frac{(q' + q^2 + 2zq)^2 + 2B}{2q\{q' + q^2 + 2zq - 2(A - 1)\}}, \quad (3.13b)$$

respectively, where

$$(A_5, B_5) = (A + 2, B), \quad (A_7, B_7) = (A - 2, B). \quad (3.13c)$$

We remark that  $\mathcal{R}_5$  and  $\mathcal{R}_7$  are the transformations  $\mathcal{T}_+$  and  $\mathcal{T}_-$ , respectively, given by Murata [56].

### 3.3 Special function solutions

The Painlevé equations  $P_{II}$ – $P_{VI}$  possess hierarchies of solutions expressible in terms of classical special functions, for special values of the parameters through an associated Riccati equation,

$$\frac{dq}{dz} = f_2(z)q^2 + f_1(z)q + f_0(z), \quad (3.14)$$

where  $f_2(z)$ ,  $f_1(z)$  and  $f_0(z)$  are rational functions. Hierarchies of solutions, which are often referred to as “one-parameter solutions” (since they have one arbitrary constant), are generated from “seed solutions” derived from the Riccati equation



using the Bäcklund transformations given in §3.2. Furthermore, as for the rational solutions, these special function solutions are often expressed in the form of determinants.

Solutions of  $P_{II}$ – $P_{VI}$  are expressed in terms of special functions as follows (see [17, 39, 54], and the references therein): for  $P_{II}$  in terms of Airy functions  $Ai(z)$  and  $Bi(z)$ ; for  $P_{III}$  in terms of Bessel functions  $J_\nu(z)$  and  $Y_\nu(z)$ ; for  $P_{IV}$  in terms of parabolic cylinder functions  $D_\nu(z)$ ; for  $P_V$  in terms of confluent hypergeometric functions  ${}_1F_1(a; c; z)$  (equivalently Kummer functions  $M(a, b, z)$  and  $U(a, b, z)$  or Whittaker functions  $M_{\kappa, \mu}(z)$  and  $W_{\kappa, \mu}(z)$ ); and for  $P_{VI}$  in terms of hypergeometric functions  ${}_2F_1(a, b; c; z)$ . Some classical orthogonal polynomials arise as particular cases of these special function solutions and thus yield rational solutions of the associated Painlevé equations: for  $P_{III}$  and  $P_V$  in terms of associated Laguerre polynomials  $L_k^{(m)}(z)$ ; for  $P_{IV}$  in terms of Hermite polynomials  $H_n(z)$ ; and for  $P_{VI}$  in terms of Jacobi polynomials  $P_n^{(\alpha, \beta)}(z)$ .

Special function solutions of  $P_{IV}$  (1.2) are expressed in terms of parabolic cylinder functions.

**Theorem 3.4.**  $P_{IV}$  (1.2) has solutions expressible in terms of parabolic cylinder functions if and only if either

$$B = -2(2n + 1 + \varepsilon A)^2, \quad (3.15)$$

or

$$B = -2n^2, \quad (3.16)$$

with  $n \in \mathbb{Z}$  and  $\varepsilon = \pm 1$ .

*Proof.* See [36, 38, 39, 40, 49, 50]. □

For  $P_{IV}$  (1.2) the associated Riccati equation is

$$\frac{dq}{dz} = \varepsilon(q^2 + 2zq) + 2\nu, \quad \varepsilon^2 = 1, \quad (3.17)$$

with  $P_{IV}$  parameters  $A = -\varepsilon(\nu + 1)$  and  $B = -2\nu^2$ . Letting  $w(z) = \frac{d}{dz} \ln \varphi_\nu(z)$  in (3.17) yields

$$\frac{d^2 \varphi_\nu}{dz^2} - 2\varepsilon z \frac{d\varphi_\nu}{dz} + 2\varepsilon \nu \varphi_\nu = 0. \quad (3.18)$$

The solution of this equation depends on whether  $\nu \in \mathbb{Z}$  or  $\nu \notin \mathbb{Z}$ , which we now summarize.

(i) If  $\nu \notin \mathbb{Z}$  then equation (3.18) has solutions

$$\varphi_\nu(z; \varepsilon) = \begin{cases} \{C_1 D_\nu(\sqrt{2}z) + C_2 D_\nu(-\sqrt{2}z)\} \exp\left(\frac{1}{2}z^2\right), & \text{if } \varepsilon = 1, \\ \{C_1 D_{-\nu-1}(\sqrt{2}z) + C_2 D_{-\nu-1}(-\sqrt{2}z)\} \exp\left(-\frac{1}{2}z^2\right), & \text{if } \varepsilon = -1, \end{cases} \quad (3.19)$$

with  $C_1$  and  $C_2$  arbitrary constants, where  $D_\nu(\zeta)$  is the *parabolic cylinder function* which satisfies

$$\frac{d^2 D_\nu}{d\zeta^2} = \left(\frac{1}{4}\zeta^2 - \nu - \frac{1}{2}\right) D_\nu, \quad (3.20)$$

and the boundary condition

$$D_\nu(\zeta) \sim \zeta^\nu \exp\left(-\frac{1}{4}\zeta^2\right), \quad \text{as } \zeta \rightarrow +\infty.$$

(ii) If  $\nu = 0$  then equation (3.18) has the solutions

$$\varphi_0(z; \varepsilon) = \begin{cases} C_1 + C_2 \operatorname{erfi}(z), & \text{if } \varepsilon = 1, \\ C_1 + C_2 \operatorname{erfc}(z), & \text{if } \varepsilon = -1, \end{cases} \quad (3.21)$$

with  $C_1$  and  $C_2$  arbitrary constants, where  $\operatorname{erfc}(z)$  is the *complementary error function* and  $\operatorname{erfi}(z)$  is the *imaginary error function*, respectively defined by

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2) dt, \quad \operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(t^2) dt. \quad (3.22)$$

(iii) If  $\nu = m$ , for  $m \geq 1$ , then equation (3.18) has the solutions

$$\varphi_m(z; \varepsilon) = \begin{cases} C_1 H_m(z) + C_2 \exp(z^2) \frac{d^m}{dz^m} \{ \operatorname{erfi}(z) \exp(-z^2) \}, & \text{if } \varepsilon = 1, \\ C_1 (-i)^m H_m(iz) + C_2 \exp(-z^2) \frac{d^m}{dz^m} \{ \operatorname{erfc}(z) \exp(z^2) \}, & \text{if } \varepsilon = -1, \end{cases} \quad (3.23)$$

with  $C_1$  and  $C_2$  arbitrary constants, where  $H_m(z)$  is the *Hermite polynomial* defined by

$$H_m(z) = (-1)^m \exp(z^2) \frac{d^m}{dz^m} \exp(-z^2). \quad (3.24)$$

(iv) If  $\nu = -m$ , for  $m \geq 1$ , then equation (3.18) has the solutions

$$\varphi_{-m}(z; \varepsilon) = \begin{cases} C_1 (-i)^{m-1} H_{m-1}(iz) \exp(z^2) + C_2 \frac{d^{m-1}}{dz^{m-1}} \{ \operatorname{erfc}(z) \exp(z^2) \}, & \text{if } \varepsilon = 1, \\ C_1 H_{m-1}(z) \exp(-z^2) + C_2 \frac{d^{m-1}}{dz^{m-1}} \{ \operatorname{erfi}(z) \exp(-z^2) \}, & \text{if } \varepsilon = -1, \end{cases} \quad (3.25)$$

with  $C_1$  and  $C_2$  arbitrary constants.

If  $\varphi_\nu(z; \varepsilon)$  is a solution of (3.18), then the “seed solutions” of  $P_{IV}$  (1.2) are given by

$$q(z; -\varepsilon(\nu + 1), -2\nu^2) = -\varepsilon \frac{d}{dz} \ln \varphi_\nu(z; \varepsilon), \quad q(z; -\varepsilon\nu, -2(\nu + 1)^2) = -2z + \varepsilon \frac{d}{dz} \ln \varphi_\nu(z; \varepsilon).$$

Hierarchies of special function solutions can be generated from these solutions using the Bäcklund transformations given in §3.2. However there is an alternative approach.

Determinantal representations of special function solutions for  $P_{IV}$  (1.2) and  $S_{IV}$  (3.8) are discussed in the following theorem.

**Theorem 3.5.** *Let  $\tau_{n,\nu}(z; \varepsilon)$  be given by*

$$\tau_{n,\nu}(z; \varepsilon) = \mathcal{W} \left( \varphi_\nu(z; \varepsilon), \frac{d\varphi_\nu}{dz}(z; \varepsilon), \dots, \frac{d^{n-1}\varphi_\nu}{dz^{n-1}}(z; \varepsilon) \right), \quad n \geq 1, \quad (3.26)$$

with  $\tau_{0,\nu}(z; \varepsilon) = 1$ , where  $\varphi_\nu(z; \varepsilon)$  is a solution of (3.18) and  $\mathcal{W}(\varphi_1, \varphi_2, \dots, \varphi_n)$  is the Wronskian. Then for  $n \geq 0$ , special function solutions of  $P_{IV}$  (1.2) are given by

$$q_{n,\nu}^{[1]}(z; A_{n,\nu}^{[1]}, B_{n,\nu}^{[1]}) = -2z + \varepsilon \frac{d}{dz} \ln \frac{\tau_{n+1,\nu}(z; \varepsilon)}{\tau_{n,\nu}(z; \varepsilon)}, \quad A_{n,\nu}^{[1]} = \varepsilon(2n - \nu), \quad B_{n,\nu}^{[1]} = -2(\nu + 1)^2, \quad (3.27a)$$

$$q_{n,\nu}^{[2]}(z; A_{n,\nu}^{[2]}, B_{n,\nu}^{[2]}) = \varepsilon \frac{d}{dz} \ln \frac{\tau_{n,\nu}(z; \varepsilon)}{\tau_{n,\nu+1}(z; \varepsilon)}, \quad A_{n,\nu}^{[2]} = \varepsilon(2\nu - n), \quad B_{n,\nu}^{[2]} = -2(n + 1)^2, \quad (3.27b)$$

$$q_{n,\nu}^{[3]}(z; A_{n,\nu}^{[3]}, B_{n,\nu}^{[3]}) = \varepsilon \frac{d}{dz} \ln \frac{\tau_{n,\nu+1}(z; \varepsilon)}{\tau_{n+1,\nu}(z; \varepsilon)}, \quad A_{n,\nu}^{[3]} = -\varepsilon(n + \nu), \quad B_{n,\nu}^{[3]} = -2(\nu - n + 1)^2, \quad (3.27c)$$

and special function solutions of  $S_{IV}$  (3.8) are given by

$$\sigma_{n,\nu}^{[1]}(z; \vartheta_0, \vartheta_\infty) = \frac{d}{dz} \ln \tau_{n,\nu}(z; \varepsilon), \quad \vartheta_0^{[1]} = \varepsilon(\nu - n + 1), \quad \vartheta_\infty^{[1]} = -\varepsilon n, \quad (3.28a)$$

$$\sigma_{n,\nu}^{[2]}(z; \vartheta_0, \vartheta_\infty) = \frac{d}{dz} \ln \tau_{n,\nu}(z; \varepsilon) - 2\varepsilon n z, \quad \vartheta_0^{[2]} = \varepsilon n, \quad \vartheta_\infty^{[2]} = \varepsilon(\nu + 1), \quad (3.28b)$$

$$\sigma_{n,\nu}^{[3]}(z; \vartheta_0, \vartheta_\infty) = \frac{d}{dz} \ln \tau_{n,\nu}(z; \varepsilon) + 2\varepsilon(\nu - n + 1)z, \quad \vartheta_0^{[3]} = -\varepsilon(\nu + 1), \quad \vartheta_\infty^{[3]} = -\varepsilon(\nu - n + 1), \quad (3.28c)$$

*Proof.* See Okamoto [60]; also Forrester and Witte [32]. □

## 4 Semi-classical Laguerre weight

In this section we consider monic orthogonal polynomials  $P_n(x; t)$ , for  $n \in \mathbb{N}$ , with respect to the semi-classical Laguerre weight (1.1), where these polynomials satisfy the three-term recurrence relation (2.11), i.e.

$$xP_n(x; t) = P_{n+1}(x; t) + \alpha_n(t)P_n(x; t) + \beta_n(t)P_{n-1}(x; t), \quad (4.1)$$

Boelen and van Assche [8, Theorem 1.1] prove the following theorem.

**Theorem 4.1.** Let  $\alpha_n(t)$  and  $\beta_n(t)$  be the coefficients in the recurrence relation (4.1) associated with the semi-classical Laguerre weight (1.1). Then the quantities

$$x_n = \frac{\sqrt{2}}{t - 2\alpha_n}, \quad y_n = 2\beta_n - n - \frac{1}{2}\lambda, \quad (4.2)$$

satisfy the discrete system

$$x_{n-1}x_n = \frac{y_n + n + \frac{1}{2}\lambda}{y_n^2 - \frac{1}{4}\lambda^2}, \quad y_n + y_{n+1} = \frac{1}{x_n} \left( \frac{t}{\sqrt{2}} - \frac{1}{x_n} \right). \quad (4.3)$$

Boelen and van Assche [8] also show that the system (4.3) can be obtained from an asymmetric discrete  $P_{IV}$  equation by a limiting process. However, from our point of view, it is more convenient to have the discrete system satisfied by  $\alpha_n$  and  $\beta_n$ , which is given in the following Lemma.

**Lemma 4.2.** The coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (4.1) associated with the semi-classical Laguerre weight (1.1) satisfy the discrete system

$$(2\alpha_n - t)(2\alpha_{n-1} - t) = \frac{(2\beta_n - n)(2\beta_n - n - \lambda)}{\beta_n}, \quad (4.4a)$$

$$2\beta_n + 2\beta_{n+1} + \alpha_n(2\alpha_n - t) = 2n + \lambda + 1. \quad (4.4b)$$

*Proof.* Substituting (4.2) into (4.3) yields the discrete system (4.4).  $\square$

Since the semi-classical Laguerre weight (1.1) has the form  $\omega_0(x) \exp(xt)$  and the moments are finite for all  $t \in \mathbb{R}$ , with  $t$  a parameter, then the coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (4.1) satisfy the Toda system, recall Theorem 2.4.

We are now in a position to prove the relationship between the coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (4.1) associated with the semi-classical Laguerre weight (1.1) and solutions of  $P_{IV}$  (1.2).

**Theorem 4.3.** The coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (4.1) associated with the semi-classical Laguerre weight (1.1) are given by

$$\alpha_n(t) = \frac{1}{2}q_n(z) + \frac{1}{2}t, \quad (4.5a)$$

$$\beta_n(t) = -\frac{1}{8}\frac{dq_n}{dz} - \frac{1}{8}q_n^2(z) - \frac{1}{4}zq_n(z) + \frac{1}{2}n + \frac{1}{4}\lambda, \quad (4.5b)$$

with  $z = \frac{1}{2}t$ , where  $q_n(z)$  satisfies

$$\frac{d^2q_n}{dz^2} = \frac{1}{2q_n} \left( \frac{dq_n}{dz} \right)^2 + \frac{3}{2}q_n^3 + 4zq_n^2 + 2(z^2 - 2n - \lambda - 1)q_n - \frac{2\lambda^2}{q_n}, \quad (4.6)$$

which is  $P_{IV}$  (1.2), with parameters

$$(A, B) = (2n + \lambda + 1, -2\lambda^2). \quad (4.7)$$

*Proof.* Solving the discrete system (4.4) for  $\alpha_{n-1}$  and  $\beta_{n+1}$  yields

$$\alpha_{n-1} = \frac{1}{2}t + \frac{(2\beta_n - n)(2\beta_n - n - \lambda)}{2(2\alpha_n - t)\beta_n},$$

$$\beta_{n+1} = -\beta_n - \frac{1}{2}(2n + \lambda + 1) - \alpha_n(\alpha_n - \frac{1}{2}t),$$

and then substituting these into (2.17) gives

$$\frac{d\alpha_n}{dt} = -\alpha_n(\alpha_n - \frac{1}{2}t) - 2\beta_n + \frac{1}{2}(2n + \lambda + 1), \quad (4.8a)$$

$$\frac{d\beta_n}{dt} = (\alpha_n - \frac{1}{2}t)\beta_n - \frac{(2\beta_n - n)(2\beta_n - n - \lambda)}{2(2\alpha_n - t)}. \quad (4.8b)$$

Solving (4.8a) for  $\beta_n$  yields

$$\beta_n = \frac{1}{2}\frac{d\alpha_n}{dt} + \frac{1}{2}\alpha_n(\alpha_n - \frac{1}{2}t) - \frac{1}{4}(2n + \lambda + 1), \quad (4.9)$$

and then substituting this into (4.8b) yields

$$\frac{d^2\alpha_n}{dt^2} = \frac{1}{2\alpha_n - t} \left( \frac{d\alpha_n}{dt} - \frac{1}{2} \right)^2 + \frac{3}{2}\alpha_n^3 - \frac{5}{4}t\alpha_n^2 + \frac{1}{4}(t^2 - 4n - 2 - 2\lambda)\alpha_n + \frac{1}{4}t(2n + \lambda + 1) - \frac{\lambda^2}{4(2\alpha_n - t)},$$

Making the transformation (4.5a) in this equation yields equation (4.6), which is  $P_{IV}$  (1.2) with parameters given by (4.7). Finally making the transformation (4.5a) in (4.9) yields (4.5b), as required.  $\square$

**Remarks 4.4.**

1. Filipuk, van Assche and Zhang [23], who considered orthonormal polynomials rather than monic orthogonal polynomials, proved the result (4.5a) for  $\alpha_n(t)$ . However Filipuk, van Assche and Zhang [23] did not give an explicit expression for  $\alpha_n(t)$ , which we do below.
2. From Theorem 3.4 we see that the parameters (4.7) satisfy (3.15) with  $\varepsilon = -1$ , and therefore satisfy the condition given in Theorem 3.4 for  $P_{IV}$  to have solutions expressible in terms of parabolic cylinder functions.
3. If  $q_n$  is a solution of equation (4.6) then the solutions  $q_{n+1}$  and  $q_{n-1}$  are given by

$$q_{n+1} = \frac{(q'_n - q_n^2 - 2zq_n)^2 - 4\lambda^2}{2q_n(q'_n - q_n^2 - 2zq_n + 4n + 2\lambda + 4)}, \quad q_{n-1} = -\frac{(q'_n + q_n^2 + 2zq_n)^2 - 4\lambda^2}{2q_n(q'_n + q_n^2 + 2zq_n - 4n - 2\lambda)},$$

where  $' \equiv d/dz$ , which are special cases of the Schlesinger transformations  $\mathcal{R}_5$  (3.13a) and  $\mathcal{R}_7$  (3.13b), respectively.

4. From Theorem 3.5, we see that the parabolic cylinder function solutions of equation (4.6) are given by

$$q_n(z) = -2z + \frac{d}{dz} \ln \frac{\tau_{n+1,\lambda}(z)}{\tau_{n,\lambda}(z)}, \quad (4.10)$$

where

$$\tau_{n,\lambda}(z) = \mathcal{W} \left( \psi_\lambda, \frac{d\psi_\lambda}{dz}, \dots, \frac{d^{n-1}\psi_\lambda}{dz^{n-1}} \right), \quad \tau_{0,\lambda}(z) = 1, \quad (4.11)$$

and  $\psi_\lambda(z)$  satisfies

$$\frac{d^2\psi_\lambda}{dz^2} - 2z \frac{d\psi_\lambda}{dz} - 2(\lambda + 1)\psi_\lambda = 0, \quad (4.12)$$

which is equation (3.18) with  $\nu = -\lambda - 1$  and  $\varepsilon = 1$ . Equation (4.12) has general solution

$$\psi_\lambda(z) = \begin{cases} \{C_1 D_{-\lambda-1}(\sqrt{2}z) + C_2 D_{-\lambda-1}(-\sqrt{2}z)\} \exp\left(\frac{1}{2}z^2\right), & \text{if } \lambda \notin \mathbb{N}, \\ C_1(-i)^m H_m(iz) \exp(z^2) + C_2 \frac{d^m}{dz^m} \{\operatorname{erfc}(z) \exp(z^2)\}, & \text{if } \lambda = m \in \mathbb{N}, \end{cases} \quad (4.13)$$

with  $C_1$  and  $C_2$  arbitrary constants, where  $D_\nu(\zeta)$  is the parabolic cylinder function,  $H_m(\zeta)$  the Hermite polynomial (3.24), and  $\operatorname{erfc}(z)$  the complementary error function (3.22).

The system (4.8) satisfied by the recurrence coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  is equivalent to the Hamiltonian system (3.4) associated with  $P_{IV}$ , as shown in the following Theorem.

**Theorem 4.5.** *The system (4.8) is equivalent to the Hamiltonian system (3.4) associated with  $P_{IV}$ .*

*Proof.* If in the system (4.8) we make the transformation

$$\alpha_n(t) = \frac{1}{2}q_n(z) + \frac{1}{2}t, \quad \beta_n(t) = -\frac{1}{2}q_n(z)p_n(z) + \frac{1}{2}(n + \lambda), \quad z = \frac{1}{2}t,$$

then  $q_n(z)$  and  $p_n(z)$  satisfy the system

$$\frac{dq_n}{dz} = 4q_n p_n - q_n^2 - 2zq_n - 2\lambda, \quad (4.14a)$$

$$\frac{dp_n}{dz} = -2p_n^2 + 2p_n q_n + 2zp_n - n - \lambda, \quad (4.14b)$$

which is the system (3.4) with  $\vartheta_0 = \lambda$  and  $\vartheta_\infty = \lambda + n$ . Conversely making the transformation

$$q_n(z) = 2\alpha_n(t) - t, \quad p_n(z) = -\frac{2\beta_n(t) - n - \lambda}{2\alpha_n(t) - t}, \quad t = 2z,$$

in the system (4.14) yields the system (4.8).  $\square$

Our main objective is to obtain explicit expressions for the coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (4.1). First we derive an explicit expression for the moment  $\mu_0(t; \lambda)$ .

**Theorem 4.6.** *For the semi-classical Laguerre weight (1.1), the moment  $\mu_0(t; \lambda)$  is given by*

$$\mu_0(t; \lambda) = \begin{cases} \frac{\Gamma(\lambda + 1) \exp(\frac{1}{8}t^2)}{2^{(\lambda+1)/2}} D_{-\lambda-1}\left(-\frac{1}{2}\sqrt{2}t\right), & \text{if } \lambda \notin \mathbb{N}, \\ \frac{1}{2}\sqrt{\pi} \frac{d^m}{dt^m} \left\{ \exp\left(\frac{1}{4}t^2\right) \left[1 + \operatorname{erf}\left(\frac{1}{2}t\right)\right] \right\}, & \text{if } \lambda = m \in \mathbb{N}, \end{cases} \quad (4.15)$$

with  $D_\nu(\zeta)$  the parabolic cylinder function and  $\operatorname{erf}(z)$  the error function. Further  $\mu_0(t; \lambda)$  satisfies the equation

$$\frac{d^2\mu_0}{dt^2} - \frac{1}{2}t \frac{d\mu_0}{dt} - \frac{1}{2}(\lambda + 1)\mu_0 = 0. \quad (4.16)$$

*Proof.* The parabolic cylinder function  $D_\nu(\zeta)$ , with  $\nu \notin \mathbb{Z}$ , has the integral representation [61, §12.5(i)]

$$D_\nu(\zeta) = \frac{\exp(-\frac{1}{4}\zeta^2)}{\Gamma(-\nu)} \int_0^\infty s^{-\nu-1} \exp(-\frac{1}{2}s^2 - \zeta s) ds, \quad \Re(\nu) < 0.$$

For the semi-classical Laguerre weight (1.1), the moment  $\mu_0(t; \lambda)$ , with  $\lambda \notin \mathbb{N}$ , is given by

$$\begin{aligned} \mu_0(t; \lambda) &= \int_0^\infty x^\lambda \exp(-x^2 + xt) dx \\ &= 2^{-(\lambda+1)/2} \int_0^\infty s^\lambda \exp\left(-\frac{1}{2}s^2 + \frac{1}{2}\sqrt{2}ts\right) ds \\ &= \frac{\Gamma(\lambda + 1) \exp(\frac{1}{8}t^2)}{2^{(\lambda+1)/2}} D_{-\lambda-1}\left(-\frac{1}{2}\sqrt{2}t\right) \end{aligned}$$

as required. If  $m \in \mathbb{N}$ , then the parabolic cylinder function  $D_{-m-1}(\zeta)$  is given by

$$D_{-m-1}(\zeta) = \sqrt{\frac{\pi}{2}} \frac{(-1)^m}{m!} \exp(-\frac{1}{4}\zeta^2) \frac{d^m}{d\zeta^m} \left\{ \exp(\frac{1}{2}\zeta^2) \operatorname{erfc}\left(\frac{1}{2}\sqrt{2}\zeta\right) \right\},$$

with  $\operatorname{erfc}(z)$  the complementary error function [61, §12.7(ii)]. Since  $\operatorname{erfc}(-z) = 1 + \operatorname{erf}(z)$ , then  $\mu_0(t; m)$ , with  $m \in \mathbb{N}$ , is given by

$$\mu_0(t; m) = \frac{1}{2}\sqrt{\pi} \frac{d^m}{dt^m} \left\{ \exp\left(\frac{1}{4}t^2\right) \left[1 + \operatorname{erf}\left(\frac{1}{2}t\right)\right] \right\},$$

as required. Further, the parabolic cylinder function  $D_\nu(\zeta)$  satisfies equation (3.20) and so from (4.15) it follows that the moment  $\mu_0(t; \lambda)$  satisfies equation (4.16), as required.  $\square$

**Corollary 4.7.** *If  $\mu_0(t; \lambda)$  is given by (4.15) and  $\varphi_\nu(z; \varepsilon)$  by (3.19), then*

$$\mu_0(t; \lambda) = \varphi_{-\lambda-1}\left(\frac{1}{2}t; 1\right), \quad (4.17)$$

with  $C_1 = 0$  and  $C_2 = \Gamma(\lambda + 1)/2^{(\lambda+1)/2}$ .

*Proof.* The result is easily shown by comparing (4.15) and (3.19).  $\square$

Having obtained an explicit expression for  $\mu_0$  we can now derive explicit expressions for the Hankel determinant  $\Delta_n(t)$  and the coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (4.1).

**Theorem 4.8.** *The Hankel determinant  $\Delta_n(t)$  is given by*

$$\Delta_n(t) = \mathcal{W} \left( \mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}} \right), \quad (4.18)$$

with  $\mu_0$  given by (4.15).

*Proof.* This is an immediate consequence of Theorem 2.1.  $\square$

**Theorem 4.9.** *The coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (4.1) associated with monic polynomials orthogonal with respect to the semi-classical Laguerre weight (1.1) are given by*

$$\alpha_n(t) = \frac{d}{dt} \ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)}, \quad \beta_n(t) = \frac{d^2}{dt^2} \ln \Delta_n(t), \quad (4.19)$$

where  $\Delta_n(t)$  is the Hankel determinant given by (4.18), with  $\mu_0$  given by (4.15).

*Proof.* This is an immediate consequence of Theorems 2.1 and 2.3.  $\square$

Furthermore we can relate the Hankel determinant  $\Delta_n(t)$  given by (4.18) to the  $\tau$ -function  $\tau_{n,\nu}(z; \varepsilon)$  given by (3.26).

**Theorem 4.10.** *If  $\Delta_n(t)$  is given by (4.18) and  $\tau_{n,\nu}(z; \varepsilon)$  by (3.26), with*

$$\varphi_{-\lambda-1}(z) = \frac{\Gamma(\lambda+1) \exp\left(\frac{1}{2}z^2\right)}{2^{(\lambda+1)/2}} D_{-\lambda-1}\left(-\sqrt{2}z\right),$$

then

$$\Delta_n(t) = \frac{\tau_{n,-\lambda-1}(z; 1)}{2^{n(n-1)}} \Big|_{z=t/2}. \quad (4.20)$$

*Proof.* The result is easily shown by comparing (4.18) and (3.26).  $\square$

**Theorem 4.11.** *The function  $S_n(t) = \frac{d}{dt} \ln \Delta_n(t)$ , with  $\Delta_n(t)$  given by (4.18), satisfies the second-order, second-degree equation*

$$4 \left( \frac{d^2 S_n}{dt^2} \right)^2 - \left( t \frac{dS_n}{dt} - S_n \right)^2 + 4 \frac{dS_n}{dt} \left( 2 \frac{dS_n}{dt} - n \right) \left( 2 \frac{dS_n}{dt} - n - \lambda \right) = 0. \quad (4.21)$$

*Proof.* Setting  $\nu = -\lambda - 1$  and  $\varepsilon = 1$  in (3.28c) gives

$$\sigma(z; \lambda, n + \lambda) = \frac{d}{dz} \ln \tau_{n,-\lambda-1}(z; 1) - 2(n + \lambda)z,$$

and so if  $S_n(t) = \frac{d}{dt} \ln \Delta_n(t)$  then from (4.20) we see that

$$\sigma(z; \lambda, n + \lambda) = 2S_n(t) - (n + \lambda)t, \quad z = \frac{1}{2}t. \quad (4.22)$$

Making this transformation in  $S_{IV}$  (3.8) with  $\vartheta_0 = \lambda$  and  $\vartheta_\infty = n + \lambda$ , i.e.

$$\left( \frac{d^2 \sigma}{dz^2} \right)^2 - 4 \left( z \frac{d\sigma}{dz} - \sigma \right)^2 + 4 \frac{d\sigma}{dz} \left( \frac{d\sigma}{dz} + 2\lambda \right) \left( \frac{d\sigma}{dz} + 2n + 2\lambda \right) = 0,$$

yields (4.21), as required.  $\square$

**Remark 4.12.** Differentiating (4.21) and letting  $S_n(t) = \frac{d}{dt} \ln \Delta_n(t)$  yields the fourth-order, bi-linear equation

$$\begin{aligned} \Delta_n \frac{d^4 \Delta_n}{dt^4} - 4 \frac{d^3 \Delta_n}{dt^3} \frac{d\Delta_n}{dt} + 3 \left( \frac{d^2 \Delta_n}{dt^2} \right)^2 - \left( \frac{1}{4}t^2 + 4n + 2\lambda \right) \left[ \Delta_n \frac{d^2 \Delta_n}{dt^2} - \left( \frac{d\Delta_n}{dt} \right)^2 \right] \\ + \frac{1}{4}t \Delta_n \frac{d\Delta_n}{dt} + \frac{1}{2}n(n + \lambda) \Delta_n^2 = 0, \end{aligned} \quad (4.23)$$

as is easily verified.

**Theorem 4.13.** *Suppose  $\Psi_{n,\lambda}(z)$  is given by*

$$\Psi_{n,\lambda}(z) = \mathcal{W} \left( \psi_\lambda, \frac{d\psi_\lambda}{dz}, \dots, \frac{d^{n-1}\psi_\lambda}{dz^{n-1}} \right), \quad \Psi_{0,\lambda}(z) = 1,$$

where

$$\psi_\lambda(z) = \begin{cases} D_{-\lambda-1}(-\sqrt{2}z) \exp\left(\frac{1}{2}z^2\right), & \text{if } \lambda \notin \mathbb{N}, \\ \frac{d^m}{dz^m} \{ [1 + \operatorname{erf}(z)] \exp(z^2) \}, & \text{if } \lambda = m \in \mathbb{N}, \end{cases}$$

with  $D_\nu(\zeta)$  is parabolic cylinder function and  $\operatorname{erfc}(z)$  the complementary error function (3.22). Then coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (4.1) associated with the semi-classical Laguerre weight (1.1) are given by

$$\alpha_n(t) = \frac{1}{2}q_n(z) + \frac{1}{2}t, \quad (4.24a)$$

$$\beta_n(t) = -\frac{1}{8}\frac{dq_n}{dz} - \frac{1}{8}q_n^2(z) - \frac{1}{4}zq_n(z) + \frac{1}{4}\lambda + \frac{1}{2}n, \quad (4.24b)$$

with  $z = \frac{1}{2}t$ , where

$$q_n(z) = -2z + \frac{d}{dz} \ln \frac{\Psi_{n+1,\lambda}(z)}{\Psi_{n,\lambda}(z)},$$

which satisfies  $P_{IV}$  (1.2), with parameters  $(A, B) = (2n + \lambda + 1, -2\lambda^2)$ .

In Appendix 1 we give the first few recurrence coefficients for the semi-classical Laguerre weight (1.1) and the first few monic polynomials generated using the recurrence relation (4.1).

## 5 Asymptotic expansions

In this section we derive asymptotic expansions for the the moment  $\mu_0(t; \lambda)$ , see Lemma 5.1 below, the Hankel determinant  $\Delta_n(t)$ , see Lemma 5.2 below, and the recurrence coefficients  $\alpha_n(t)$  and  $\beta_n(t)$ , see Lemma 5.3 below.

**Lemma 5.1.** *As  $t \rightarrow \infty$ , the moment  $\mu_0(t; \lambda)$  has the asymptotic expansion*

$$\mu_0(t; \lambda) \sim \sqrt{\pi} \left(\frac{1}{2}t\right)^\lambda \exp\left(\frac{1}{4}t^2\right) \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1) n! t^{2n}}. \quad (5.1)$$

*Proof.* Since the parabolic cylinder function  $D_\nu(\zeta)$  has the asymptotic expansion

$$D_\nu(\zeta) \sim \frac{\sqrt{2\pi}(-1)^{\nu+1}}{\Gamma(-\nu)\zeta^{\nu+1}} \exp\left(\frac{1}{4}\zeta^2\right) \sum_{n=0}^{\infty} \frac{(\nu+1)_{2n}}{n! (2\zeta^2)^n}, \quad \text{as } \zeta \rightarrow -\infty,$$

with  $(\beta)_n = \Gamma(\beta+n)/\Gamma(\beta)$  the Pochhammer symbol, then

$$\begin{aligned} \mu_0(t) &= \frac{\Gamma(\lambda+1) \exp\left(\frac{1}{8}t^2\right)}{2^{(\lambda+1)/2}} D_{-\lambda-1}\left(-\frac{1}{2}\sqrt{2}t\right) \\ &\sim \frac{\Gamma(\lambda+1) \exp\left(\frac{1}{8}t^2\right)}{2^{(\lambda+1)/2}} \frac{\sqrt{2\pi} t^\lambda \exp\left(\frac{1}{8}t^2\right)}{\Gamma(\lambda+1) 2^{\lambda/2}} \sum_{n=0}^{\infty} \frac{(-\lambda)_{2n}}{n! t^{2n}} \\ &= \sqrt{\pi} \left(\frac{1}{2}t\right)^\lambda \exp\left(\frac{1}{4}t^2\right) \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1) n! t^{2n}}, \end{aligned}$$

as required, since

$$(-\lambda)_{2n} = \frac{\Gamma(2n-\lambda)}{\Gamma(-\lambda)} = \lambda(\lambda-1)\dots(\lambda-2n+1) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}.$$

□

**Lemma 5.2.** *As  $t \rightarrow \infty$ , the Hankel determinant  $\Delta_n(t)$  has the asymptotic expansion*

$$\Delta_n(t) = c_n \pi^{n/2} \left(\frac{1}{2}t\right)^{n\lambda} \exp\left(\frac{1}{4}nt^2\right) \left\{ 1 + \frac{n\lambda(\lambda-n)}{t^2} + \mathcal{O}(t^{-4}) \right\}, \quad (5.2)$$

with  $c_n$  a constant, and  $S_n(t)$  has the asymptotic expansion

$$S_n(t) = \frac{nt}{2} + \frac{n\lambda}{t} + \frac{2n\lambda(n-\lambda)}{t^3} + \mathcal{O}(t^{-5}). \quad (5.3)$$

*Proof.* To prove (5.2) we shall use Mathematical induction. Since  $\Delta_n$  satisfies the Toda equation (2.13) then

$$\Delta_{n+1} = \frac{1}{\Delta_{n-1}} \left\{ \Delta_n \frac{d^2 \Delta_n}{dt^2} - \left( \frac{d\Delta_n}{dt} \right)^2 \right\}. \quad (5.4)$$

By definition  $\Delta_0 = 1$  and from (5.1)

$$\Delta_1 = \mu_0 = \sqrt{\pi} \left( \frac{1}{2} t \right)^\lambda \exp \left( \frac{1}{4} t^2 \right) \left\{ 1 + \frac{\lambda(\lambda-1)}{t^2} + \mathcal{O}(t^{-4}) \right\}. \quad (5.5)$$

as  $t \rightarrow \infty$ , and so (5.4) with  $n = 1$  gives

$$\Delta_2 = \left\{ \Delta_1 \frac{d^2 \Delta_1}{dt^2} - \left( \frac{d\Delta_1}{dt} \right)^2 \right\} = \frac{1}{2} \pi \left( \frac{1}{2} t \right)^{2\lambda} \exp \left( \frac{1}{2} n t^2 \right) \left\{ 1 + \frac{2\lambda(\lambda-2)}{t^2} + \mathcal{O}(t^{-4}) \right\},$$

as  $t \rightarrow \infty$ . Assuming (5.2) then

$$\left\{ \Delta_n \frac{d^2 \Delta_n}{dt^2} - \left( \frac{d\Delta_n}{dt} \right)^2 \right\} = \frac{1}{2} n c_n^2 \pi^n \left( \frac{1}{2} t \right)^{2n\lambda} \exp \left( \frac{1}{2} n t^2 \right) \left\{ 1 + \frac{2\lambda(n\lambda - n^2 - 1)}{t^2} + \mathcal{O}(t^{-4}) \right\},$$

as  $t \rightarrow \infty$ , and so

$$\begin{aligned} \Delta_{n+1} &= \frac{1}{\Delta_{n-1}} \left\{ \Delta_n \frac{d^2 \Delta_n}{dt^2} - \left( \frac{d\Delta_n}{dt} \right)^2 \right\} \\ &= \frac{n c_n^2}{2 c_{n-1}} \pi^{(n+1)/2} \left( \frac{1}{2} t \right)^{(n+1)\lambda} \exp \left\{ \frac{1}{4} (n+1) t^2 \right\} \\ &\quad \times \left\{ 1 + \frac{2\lambda(n\lambda - n^2 - 1)}{t^2} + \mathcal{O}(t^{-4}) \right\} \left\{ 1 - \frac{(n-1)\lambda(\lambda - n + 1)}{t^2} + \mathcal{O}(t^{-4}) \right\} \\ &= c_{n+1} \pi^{(n+1)/2} \left( \frac{1}{2} t \right)^{(n+1)\lambda} \exp \left\{ \frac{1}{4} (n+1) t^2 \right\} \left\{ 1 + \frac{(n+1)\lambda(\lambda - n - 1)}{t^2} + \mathcal{O}(t^{-4}) \right\} \end{aligned}$$

as  $t \rightarrow \infty$ , where  $c_{n+1} = \frac{1}{2} n c_n^2 / c_{n-1}$ , as required. Solving the recurrence relation

$$c_{n+1} c_{n-1} = \frac{1}{2} n c_n^2, \quad c_0 = 1, \quad c_1 = 1,$$

gives

$$c_n = \frac{1}{2^{n(n-1)/2}} \prod_{k=0}^{n-1} k!.$$

Since  $S_n = \frac{d}{dt} \ln \Delta_n$  then the asymptotic expansion (5.3) is easily derived from (5.2). □

**Lemma 5.3.** As  $t \rightarrow \infty$ , the recurrence coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  have the asymptotic expansions

$$\alpha_n(t) = \frac{t}{2} + \frac{\lambda}{t} + \frac{2\lambda(2n - \lambda + 1)}{t^3} + \mathcal{O}(t^{-5}), \quad (5.6a)$$

$$\beta_n(t) = \frac{n}{2} - \frac{n\lambda}{t^2} - \frac{6n\lambda(n - \lambda)}{t^4} + \mathcal{O}(t^{-6}). \quad (5.6b)$$

*Proof.* By definition

$$\alpha_n(t) = \frac{d}{dt} \ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)} = S_{n+1}(t) - S_n(t), \quad \beta_n(t) = \frac{d^2}{dt^2} \ln \Delta_n(t) = \frac{dS_n}{dt},$$

and so

$$\alpha_n(t) = \frac{t}{2} + \frac{\lambda}{t} + \frac{2\lambda(2n - \lambda + 1)}{t^3} + \mathcal{O}(t^{-5}), \quad \beta_n(t) = \frac{n}{2} - \frac{n\lambda}{t^2} - \frac{6n\lambda(n - \lambda)}{t^4} + \mathcal{O}(t^{-6}),$$

as  $t \rightarrow \infty$ , as required. Consequently

$$\lim_{t \rightarrow \infty} \alpha_n(t) = \frac{1}{2} t, \quad \lim_{t \rightarrow \infty} \beta_n(t) = \frac{1}{2} n.$$

□



## 6 Semi-classical Hermite weight

In this section we are concerned with the semi-classical Hermite weight

$$\omega(x; t) = |x|^\lambda \exp(-x^2 + tx), \quad x, t \in \mathbb{R}, \quad \lambda > -1, \quad (6.1)$$

which is an extension of the semi-classical Laguerre weight (1.1) to the whole real line, where we have ensured that the weight is positive by using  $|x|^\lambda$  rather than  $x^\lambda$ . Monic orthogonal polynomials associated with the semi-classical Hermite weight (6.1) satisfy the recurrence relation

$$xP_n(x; t) = P_{n+1}(x; t) + \alpha_n(t)P_n(x; t) + \beta_n(t)P_{n-1}(x; t), \quad (6.2)$$

and our interest is in obtaining explicit expressions for the coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in (6.2).

First we evaluate the moment  $\mu_0(t; \lambda)$ .

**Theorem 6.1.** *For the semi-classical Hermite weight (6.1), the moment  $\mu_0(t; \lambda)$  is given by*

$$\mu_0(t; \lambda) = \begin{cases} \frac{\Gamma(\lambda + 1) \exp(\frac{1}{8}t^2)}{2^{(\lambda+1)/2}} \left\{ D_{-\lambda-1}(-\frac{1}{2}\sqrt{2}t) + D_{-\lambda-1}(\frac{1}{2}\sqrt{2}t) \right\}, & \text{if } \lambda \notin \mathbb{N}, \\ \sqrt{\pi} \left(-\frac{1}{2}i\right)^{2m} H_{2m}(\frac{1}{2}it) \exp\left(\frac{1}{4}t^2\right), & \text{if } \lambda = 2m, \\ \sqrt{\pi} \frac{d^{2m+1}}{dt^{2m+1}} \left\{ \operatorname{erf}\left(\frac{1}{2}t\right) \exp\left(\frac{1}{4}t^2\right) \right\}, & \text{if } \lambda = 2m + 1, \end{cases} \quad (6.3)$$

with  $m \in \mathbb{N}$ , where  $D_\nu(z)$  is the parabolic cylinder function,  $H_n(z)$  is the Hermite polynomial and  $\operatorname{erf}(z)$  is the error function.

*Proof.* If  $\lambda \notin \mathbb{N}$ , then the moment  $\mu_0(t; \lambda)$  is given by

$$\begin{aligned} \mu_0(t; \lambda) &= \int_{-\infty}^{\infty} \omega(x; t) dx = \int_{-\infty}^{\infty} |x|^\lambda \exp(-x^2 + tx) dx \\ &= \int_0^{\infty} x^\lambda \exp(-x^2 + tx) dx + \int_0^{\infty} x^\lambda \exp(-x^2 - tx) dx \\ &= \frac{\Gamma(\lambda + 1) \exp(\frac{1}{8}t^2)}{2^{(\lambda+1)/2}} \left\{ D_{-\lambda-1}(-\frac{1}{2}\sqrt{2}t) + D_{-\lambda-1}(\frac{1}{2}\sqrt{2}t) \right\}, \end{aligned} \quad (6.4)$$

as required. If  $\lambda = 2m$ , with  $m \in \mathbb{N}$ , then

$$\mu_0(t; 2m) = \int_{-\infty}^{\infty} x^{2m} \exp(-x^2 + tx) dx = \sqrt{\pi} \left(-\frac{1}{2}i\right)^{2m} H_{2m}(\frac{1}{2}it) \exp\left(\frac{1}{4}t^2\right), \quad (6.5)$$

as required, since the Hermite polynomial,  $H_n(z)$ , has the integral representation

$$H_m(z) = \frac{2^m}{\sqrt{\pi}} \int_{-\infty}^{\infty} (z + ix)^m \exp(-x^2) dx.$$

Finally if  $\lambda = 2m + 1$ , with  $m \in \mathbb{N}$ , then

$$\begin{aligned} \mu_0(t; 2m + 1) &= \int_{-\infty}^{\infty} x^{2m+1} |x| \exp(-x^2 + tx) dx \\ &= \frac{d^{2m}}{dt^{2m}} \left( \int_0^{\infty} x \exp(-x^2 + tx) dx + \int_0^{\infty} x \exp(-x^2 - tx) dx \right) \\ &= \frac{d^{2m+1}}{dt^{2m+1}} \left( \int_0^{\infty} \exp(-x^2 + tx) dx - \int_0^{\infty} \exp(-x^2 - tx) dx \right) \\ &= \frac{d^{2m+1}}{dt^{2m+1}} \left( \frac{1}{2}\sqrt{\pi} \{1 + \operatorname{erf}(\frac{1}{2}t)\} \exp\left(\frac{1}{4}t^2\right) - \frac{1}{2}\sqrt{\pi} \{1 - \operatorname{erf}(\frac{1}{2}t)\} \exp\left(\frac{1}{4}t^2\right) \right) \\ &= \sqrt{\pi} \frac{d^{2m+1}}{dt^{2m+1}} \left\{ \operatorname{erf}\left(\frac{1}{2}t\right) \exp\left(\frac{1}{4}t^2\right) \right\}, \end{aligned}$$

as required, since

$$\int_0^{\infty} \exp(-x^2 + tx) dx = \frac{1}{2}\sqrt{\pi} \{1 + \operatorname{erf}(\frac{1}{2}t)\} \exp\left(\frac{1}{4}t^2\right).$$

□

Next we obtain an explicit expression for the Hankel determinant  $\Delta_n(t)$ .

**Theorem 6.2.** *The Hankel determinant  $\Delta_n(t)$  is given by*

$$\Delta_n(t) = \mathcal{W} \left( \mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}} \right), \quad (6.6)$$

where  $\mu_0(t; \lambda)$  is given by (6.3).

*Proof.* By definition the moment  $\mu_k(t; \lambda)$  is given by

$$\begin{aligned} \mu_k(t; \lambda) &= \int_{-\infty}^{\infty} x^k |x|^\lambda \exp(-x^2 + tx) dx \\ &= \frac{d^k}{dt^k} \left( \int_{-\infty}^{\infty} |x|^\lambda \exp(-x^2 + tx) dx \right) = \frac{d^k \mu_0}{dt^k}, \end{aligned}$$

and so we obtain

$$\Delta_n(t) = \det \left[ \mu_{j+k}(t) \right]_{j,k=0}^n \equiv \mathcal{W} \left( \mu_0, \frac{d\mu_0}{dt}, \dots, \frac{d^{n-1}\mu_0}{dt^{n-1}} \right),$$

as required.  $\square$

Finally we obtain explicit expressions for the coefficients  $\alpha_n(t)$  and  $\beta_n(t)$ .

**Theorem 6.3.** *The coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (6.2) associated with monic polynomials orthogonal with respect to the semi-classical Hermite weight (6.1) are given by*

$$\alpha_n(t) = \frac{d}{dt} \ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)}, \quad \beta_n(t) = \frac{d^2}{dt^2} \ln \Delta_n(t), \quad (6.7)$$

where  $\Delta_n(t)$  is the Hankel determinant given by (6.6), with  $\mu_0(t; \lambda)$  given by (6.3).

*Proof.* This is an immediate consequence of Theorem 2.3.  $\square$

**Theorem 6.4.** *Suppose  $\tilde{\Psi}_{n,\lambda}(z)$  is given by*

$$\tilde{\Psi}_{n,\lambda}(z) = \mathcal{W} \left( \tilde{\psi}_\lambda, \frac{d\tilde{\psi}_\lambda}{dz}, \dots, \frac{d^{n-1}\tilde{\psi}_\lambda}{dz^{n-1}} \right), \quad \Psi_{0,\lambda}(z) = 1,$$

where

$$\tilde{\psi}_\lambda(z) = \begin{cases} \{D_{-\lambda-1}(\sqrt{2}z) + D_{-\lambda-1}(-\sqrt{2}z)\} \exp\left(\frac{1}{2}z^2\right), & \text{if } \lambda \notin \mathbb{N}, \\ H_{2m}(iz) \exp(z^2), & \text{if } \lambda = 2m, \quad m \in \mathbb{N}, \\ \frac{d^{2m+1}}{dz^{2m+1}} \{\operatorname{erf}(z) \exp(z^2)\}, & \text{if } \lambda = 2m+1, \quad m \in \mathbb{N}, \end{cases}$$

with  $D_\nu(\zeta)$  is parabolic cylinder function,  $H_m(\zeta)$  the Hermite polynomial (3.24), and  $\operatorname{erfc}(z)$  the complementary error function (3.22). Then coefficients  $\alpha_n(t)$  and  $\beta_n(t)$  in the recurrence relation (6.2) associated with the semi-classical Hermite weight (6.1) are given by

$$\alpha_n(t) = \frac{1}{2}q_n(z) + \frac{1}{2}t, \quad (6.8a)$$

$$\beta_n(t) = -\frac{1}{8}\frac{dq_n}{dz} - \frac{1}{8}q_n^2(z) - \frac{1}{4}zq_n(z) + \frac{1}{4}\lambda + \frac{1}{2}n, \quad (6.8b)$$

with  $z = \frac{1}{2}t$ , where

$$q_n(z) = -2z + \frac{d}{dz} \ln \frac{\tilde{\Psi}_{n+1,\lambda}(z)}{\tilde{\Psi}_{n,\lambda}(z)},$$

which satisfies  $P_{IV}$  (1.2), with parameters given by  $(A, B) = (2n + \lambda + 1, -2\lambda^2)$ .

In Appendix 2 we give the first few recurrence coefficients for the semi-classical Hermite weight (6.1), in the case when  $\lambda = 2$  (so the recurrence coefficients are rational functions of  $t$ ), and the first few monic polynomials generated using the recurrence relation (6.2).

## 7 Discussion

In this paper we have studied semi-classical Laguerre polynomials which are orthogonal polynomials that satisfy three-term recurrence relations whose coefficients depend on a parameter. We have shown that the coefficients in these recurrence relations can be expressed in terms of Wronskians of parabolic cylinder functions. These Wronskians also arise in the description of special function solutions of the fourth Painlevé equation and the second-order, second-degree equation satisfied by the associated Hamiltonian function. Further we have shown similar results hold for semi-classical Hermite polynomials. The link between the semi-classical orthogonal polynomials and the special function solutions of the Painlevé equations is the moment for the associated weight which enables the Hankel determinant to be written as a Wronskian. In our opinion, this illustrates the increasing significance of the Painlevé equations in the field of orthogonal polynomials and special functions.

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## Appendix 1. Recurrence coefficients and polynomials for the semi-classical Laguerre weight

For the semi-classical Laguerre weight the first few recurrence coefficients are given by

$$\begin{aligned}\alpha_0(t) &= \frac{1}{2}t - \frac{D_{-\lambda}(-\frac{1}{2}\sqrt{2}t)}{D_{-\lambda-1}(-\frac{1}{2}\sqrt{2}t)} \equiv \Psi_\nu(t), \\ \alpha_1(t) &= \frac{1}{2}t - \Psi_\nu(t) - \frac{\Psi_\nu(t)}{2\Psi_\nu^2(t) - t\Psi_\nu(t) - \lambda - 1}, \\ \alpha_2(t) &= \frac{1}{2}t + \frac{2\lambda + 4}{t} + \frac{\Psi_\nu(t)}{2\Psi_\nu^2(t) - t\Psi_\nu(t) - \lambda - 1}, \\ &\quad - \frac{2[(\lambda + 1)t^2 + 4(\lambda + 2)(2\lambda + 3)]\Psi_\nu^2(t) - (\lambda + 1)t[t^2 + 2(4\lambda + 9)]\Psi_\nu(t) - (\lambda + 1)^2[t^2 + 8(\lambda + 2)]}{2t[2t\Psi_\nu^3(t) - (t^2 - 4\lambda - 6)\Psi_\nu^2(t) - 3(\lambda + 1)t\Psi_\nu(t) - 2(\lambda + 1)^2]}, \\ \beta_1(t) &= -\Psi_\nu^2(t) + \frac{1}{2}t\Psi_\nu(t) + \frac{1}{2}(\lambda + 1), \\ \beta_2(t) &= -\frac{2t\Psi_\nu^3(t) - (t^2 - 4\lambda - 6)\Psi_\nu^2(t) - 3(\lambda + 1)t\Psi_\nu(t) - 2(\lambda + 1)^2}{2[\Psi_\nu^2(t) - \frac{1}{2}t\Psi_\nu(t) - \frac{1}{2}(\lambda + 1)]^2},\end{aligned}$$

and the first few monic orthogonal polynomials are given by

$$\begin{aligned}P_1(x; t) &= x - \Psi_\nu \\ P_2(x; t) &= x^2 - \frac{2t\Psi_\nu^2 - (t^2 + 2)\Psi_\nu - (\lambda + 1)t}{2[\Psi_\nu^2 - \frac{1}{2}t\Psi_\nu - \frac{1}{2}(\lambda + 1)]}x - \frac{2(\lambda + 2)\Psi_\nu^2 - (\lambda + 1)\Psi_\nu - (\lambda + 1)^2}{2[\Psi_\nu^2 - \frac{1}{2}t\Psi_\nu - \frac{1}{2}(\lambda + 1)]} \\ P_3(x; t) &= x^3 - \left\{ \frac{4(t^2 + 2\lambda + 4)\Psi_\nu^3 - 2t(t^2 - \lambda - 1)\Psi_\nu^2 - (\lambda + 1)(5t^2 + 4\lambda + 6)\Psi_\nu - 3(\lambda + 1)^2t}{2[2t\Psi_\nu^3 - (t^2 - 4\lambda - 6)\Psi_\nu^2 - 3(\lambda + 1)t\Psi_\nu - 2(\lambda + 1)^2]} \right\} x^2 \\ &\quad + \left\{ \frac{2t(t^2 + 2\lambda + 4)\Psi_\nu^3 - [t^4 + 4(2\lambda + 5)(\lambda + 2)]\Psi_\nu^2 - 2(\lambda + 1)t(t^2 - \lambda - 5)\Psi_\nu - (\lambda + 1)^2(t^2 - 4\lambda - 12)}{4[2t\Psi_\nu^3 - (t^2 - 4\lambda - 6)\Psi_\nu^2 - 3(\lambda + 1)t\Psi_\nu - 2(\lambda + 1)^2]} \right\} x \\ &\quad + \frac{2[(\lambda + 1)t^2 + 4(\lambda + 2)]\Psi_\nu^3 - (\lambda + 1)t(t^2 + 2\lambda + 8)\Psi_\nu^2 - 2(\lambda + 1)^2(t^2 + 2\lambda + 5)\Psi_\nu - (\lambda + 1)^3t}{4[2t\Psi_\nu^3 - (t^2 - 4\lambda - 6)\Psi_\nu^2 - 3(\lambda + 1)t\Psi_\nu - 2(\lambda + 1)^2]}\end{aligned}$$

## Appendix 2. Recurrence coefficients and polynomials for the semi-classical Hermite weight

For the semi-classical Hermite weight  $x^2 \exp(-x^2 + tx)$  the first few recurrence coefficients are given by

$$\begin{aligned}
\alpha_0(t) &= \frac{1}{2}t + \frac{2t}{t^2 + 2}, \\
\alpha_1(t) &= \frac{1}{2}t + \frac{4t^3}{t^4 + 12} - \frac{2t}{t^2 + 2}, \\
\alpha_2(t) &= \frac{1}{2}t + \frac{6t(t^4 + 12 - 4t^2)}{t^6 - 6t^4 + 36t^2 + 72} - \frac{4t^3}{t^4 + 12}, \\
\alpha_3(t) &= \frac{1}{2}t + \frac{8t^3(t^4 + 60 - 12t^2)}{t^8 - 16t^6 + 120t^4 + 720} - \frac{6t(t^4 + 12 - 4t^2)}{t^6 - 6t^4 + 36t^2 + 72}, \\
\alpha_4(t) &= \frac{1}{2}t + \frac{10t(t^8 + 216t^4 + 720 - 24t^6 - 480t^2)}{t^{10} - 30t^8 + 360t^6 - 1200t^4 + 3600t^2 + 7200} - \frac{8t^3(t^4 + 60 - 12t^2)}{t^8 - 16t^6 + 120t^4 + 720}, \\
\alpha_5(t) &= \frac{1}{2}t + \frac{12t^3(t^8 - 40t^6 + 600t^4 - 3360t^2 + 8400)}{t^{12} - 48t^{10} + 900t^8 - 6720t^6 + 25200t^4 + 100800} - \frac{10t(t^8 + 216t^4 + 720 - 24t^6 - 480t^2)}{t^{10} - 30t^8 + 360t^6 - 1200t^4 + 3600t^2 + 7200}, \\
\beta_1(t) &= \frac{1}{2} - \frac{2(t^2 - 2)}{(t^2 + 2)^2}, \\
\beta_2(t) &= 1 - \frac{4t^2(t^2 - 6)(t^2 + 6)}{(t^4 + 12)^2}, \\
\beta_3(t) &= \frac{3}{2} - \frac{6(t^4 - 12t^2 + 12)(t^6 + 6t^4 + 36t^2 - 72)}{(t^6 - 6t^4 + 36t^2 + 72)^2}, \\
\beta_4(t) &= 2 - \frac{8t^2(t^4 - 20t^2 + 60)(t^8 + 72t^4 - 2160)}{(t^8 - 16t^6 + 120t^4 + 720)^2}, \\
\beta_5(t) &= \frac{5}{2} - \frac{10(t^6 - 30t^4 + 180t^2 - 120)(t^{12} - 12t^{10} + 180t^8 - 480t^6 - 3600t^4 - 43200t^2 + 43200)}{(t^{10} - 30t^8 + 360t^6 - 1200t^4 + 3600t^2 + 7200)^2},
\end{aligned}$$

and the first few monic orthogonal polynomials are given by

$$\begin{aligned}
P_1(x; t) &= x - \frac{t(t^2 + 6)}{2(t^2 + 2)}, \\
P_2(x; t) &= x^2 - \frac{t(t^4 + 4t^2 + 12)}{t^4 + 12}x + \frac{t^6 + 6t^4 + 36t^2 - 72}{4(t^4 + 12)}, \\
P_3(x; t) &= x^3 - \frac{3t(t^6 - 2t^4 + 20t^2 + 120)}{2(t^6 - 6t^4 + 36t^2 + 72)}x^2 + \frac{3(t^8 + 40t^4 - 240)}{4(t^6 - 6t^4 + 36t^2 + 72)}x - \frac{t(t^8 + 72t^4 - 2160)}{8(t^6 - 6t^4 + 36t^2 + 72)}, \\
P_4(x; t) &= x^4 - \frac{2t(t^8 - 12t^6 + 72t^4 + 240t^2 + 720)}{t^8 - 16t^6 + 120t^4 + 720}x^3 + \frac{3(t^{10} - 10t^8 + 80t^6 + 1200t^2 - 2400)}{2(t^8 - 16t^6 + 120t^4 + 720)}x^2 \\
&\quad - \frac{t(t^{10} - 10t^8 + 120t^6 - 240t^4 - 1200t^2 - 7200)}{2(t^8 - 16t^6 + 120t^4 + 720)}x \\
&\quad + \frac{t^{12} - 12t^{10} + 180t^8 - 480t^6 - 3600t^4 - 43200t^2 + 43200}{16(t^8 - 16t^6 + 120t^4 + 720)}, \\
P_5(x; t) &= x^5 - \frac{5t(10080 + t^{10} + 264t^6 + 1680t^2 - 26t^8 - 336t^4)}{2(t^{10} - 30t^8 + 360t^6 - 1200t^4 + 3600t^2 + 7200)}x^4 \\
&\quad + \frac{5(t^{12} - 24t^{10} + 252t^8 - 672t^6 + 5040t^4 - 20160)}{2(t^{10} - 30t^8 + 360t^6 - 1200t^4 + 3600t^2 + 7200)}x^3 \\
&\quad - \frac{5t(t^{12} - 24t^{10} + 300t^8 - 1440t^6 + 5040t^4 - 100800)}{4(t^{10} - 30t^8 + 360t^6 - 1200t^4 + 3600t^2 + 7200)}x^2 \\
&\quad + \frac{5(t^{14} - 26t^{12} + 396t^{10} - 2520t^8 + 5040t^6 - 50400t^4 - 100800t^2 + 201600)}{16(t^{10} - 30t^8 + 360t^6 - 1200t^4 + 3600t^2 + 7200)}x \\
&\quad - \frac{t(t^{14} - 30t^{12} + 540t^{10} - 4200t^8 + 10800t^6 - 151200t^4 - 504000t^2 + 3024000)}{32(t^{10} - 30t^8 + 360t^6 - 1200t^4 + 3600t^2 + 7200)}.
\end{aligned}$$

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